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Non-canonical quantization as a source of symmetry breaking

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Abstract. With chiral theories (and gravity) in mind, we use block renormalization-group (RG) methods and RG equations to examine the proposal by de Alfaro, Fubini and Furlan that symmetry-breaking can be induced by the choice of conformal-invariant measures.

1. Introduction

With the advent of string theories there has been a realization that all *local* field theories can be understood as low-energy effective theories of something more fundamental. This has been the working practice for two decades or more in hadronic physics where chiral models have provided effective calculational tools (in both senses of 'effective') for describing the interactions of pions and hadronic matter at low energies (see, for instance, de Alfaro *et al* 1973).

In this paper we look into some questions posed by a scheme for quantizing low-energy chiral theories, proposed by de Alfaro *et al* (1981, 1983a, b, 1984) (see also Floreanini *et al* 1984) in the context of 'two-time' quantization. The main idea is that the symmetry breaking that enables the σ-field to have non-zero expectation values is due to a *non-canonical* choice of measure in the path-integral 'sum over histories'.

Changing the measure from the canonical form has a long pedigree, both for scalar theories (Klauder 1970a, 1973, 1975, 1977a, b, 1979a, b, 1981, Nouri-Moghadam *et al* 1978a, b, 1979, Ebbutt *et al* 1982, Ogielski 1983 and Gent *et al* 1986) and for gravity (Klauder 1970b, Klauder and Aslaksen 1970, Isham and Kakas 1984a, b) [also considered at some length by de Alfaro *et al*]. For scalar theories the motives have been (*a*) to give sense to perturbation theory for non-renormalizable theories (in analogy to singular potential theory) and (*b*) to evade triviality in renormalizable, but trivial, theories. (The motives in quantum gravity have been more ambitious; viz, to quantize the metric in such a way as to prohibit unitarily implementable translations of it, so as to preserve its signature. Necessarily this is a much more difficult programme, and we shall make no comments on it here.)

The approach adopted by de Alfaro *et al* is essentially that of Klauder (1981) (continued by Ogielski 1983 and Gent *et al* 1986). Consider the scalar sector of chiral theories, as exemplified by the σ-model with fields ϕ_α ($\alpha = 1, 2, 3, 4$) possessing a global $SO(4) \sim SU(2)\times SU(2)$ invariance. This invariance is broken to $SO(3) \sim SU(2)$ by the 'background'

$$\langle 0|\phi_\alpha|0\rangle = f_\pi \delta_{\alpha 4} \tag{1.1}$$

with respect to which ϕ_α decomposes as

$$\phi_\alpha = f_\pi \delta_{\alpha 4} + h_\alpha \qquad \langle 0|h_\alpha(x)|0\rangle = 0. \tag{1.2}$$

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The dimensional parameter f_π describes the low-energy coupling of pions to any hadronic system.

It is customary to take the chiral Lagrangian as

$$\mathscr{L}=\tfrac{1}{2}(\partial_\mu\phi_\alpha)^2-\tfrac{1}{2}\mu^2\phi_\alpha^2+\lambda(\phi_\alpha^2)^2. \tag{1.3}$$

If $\mu^2>0$ the density (1.3) possesses the conventional 'Mexican-hat' potential that induces symmetry breaking. However, at a formal level de Alfaro *et al* have argued that the symmetry-breaking of (1.2) is better understood as a consequence of a *non-canonical* quantization procedure imposed upon the *conformally-invariant* chiral Lagrangian density

$$\mathscr{L}_C=\tfrac{1}{2}(\partial_\mu\phi_\alpha)^2+\lambda(\phi_\alpha^2)^2 \tag{1.4}$$

with *no* dimensional parameter (and no classical symmetry-breaking).

Specifically, we can attempt to preserve conformal invariance in the quantization procedure by summing over configurations

$$Z=\int \mathrm{d}\Omega_C(\phi)\exp\left(-\int \mathrm{d}^4x\,\mathscr{L}_C\right) \tag{1.5}$$

with respect to the *conformally-invariant* measure $\mathrm{d}\Omega_C(\phi)$. This can be written (loosely) as

$$\mathrm{d}\Omega_C(\phi)=\mathrm{d}\Omega(\phi)\left(\prod_x\phi_\alpha^2(x)\right)^2 \tag{1.6}$$

where

$$\mathrm{d}\Omega(\phi)=\prod_{\alpha=1}^{4}\mathrm{d}\Omega(\phi_\alpha)=\prod_{\alpha=1}^{4}\prod_x \mathrm{d}\phi_\alpha(x) \tag{1.7}$$

is the usual canonical (translationally-invariant) measure for which $\mathrm{d}\Omega(\phi_\alpha+\Lambda_\alpha)=\mathrm{d}\Omega(\phi_\alpha)$, and Π_x denotes a formal product over all spacetime points.

A priori there is nothing quantum-mechanically inconsistent about making such a choice. In an operator formalism the change in measure (1.6) corresponds to replacing the canonical equal-time commutation relations

$$[\phi_\alpha(\boldsymbol{x},t),\Pi_\beta(\boldsymbol{y},t)]=\mathrm{i}\delta_{\alpha\beta}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \tag{1.8}$$

by the affine commutation relations (Klauder 1973, 1977, 1979a, b, Nouri-Moghadam *et al* 1978a, b, 1979 and Ebbutt *et al* 1982)

$$[\phi_\alpha(\boldsymbol{x},t),K_\beta(\boldsymbol{y},t)]=\mathrm{i}\delta_{\alpha\beta}\phi_\beta(\boldsymbol{x})\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \tag{1.9}$$

between ϕ_α and K_β, the generators of scaling transformations. Formally, $K_\beta(x)=\tfrac{1}{2}(\phi_\beta(x)\Pi_\beta(x)+\Pi_\beta(x)\phi_\beta(x))$. Since K_β is expressed via an operator product that needs renormalization to make it properly defined, equations (1.8) and (1.9) might be expected to describe different theories. However, each choice (should they be different) makes sense quantum-mechanically, both being a generalization of the single degree-of-freedom relation $[q,p]=i$.

In particular, on taking K_α, rather than Π_α, to be self-adjoint (i.e. requiring scale transformations to be unitarily implementable, rather than field translations) there is, *a priori*, no problem with unitarity. In fact, for the properly renormalized theory we shall find, embarrassingly, that changing the measure has changed nothing, and we are in the same equivalence class as the canonical theory, which is almost certainly trivial (Lüscher and Weisz 1987, 1988a, 1989 and Fröhlich 1982). This is an example of the deceptive appearance of the path integral (1.5). Only after implementing renormalization do we understand what a change of measure entails. This is second nature to many-body theorists, but less familiar to field theorists. Only for the low-energy effective theory, for which there is necessarily a unitarity bound, do we get new results. Even then we are in close correspondence with the canonical theory, as will be seen.

More prosaically, Z can be written in terms of the canonical 'measure' $\mathrm{d}\Omega$ as

$$Z = \int \mathrm{d}\Omega(\phi) \exp\left(-\int \mathrm{d}^4x\, \bar{\mathscr{L}}\right) \tag{1.10}$$

where

$$\bar{\mathscr{L}}(\phi) = \mathscr{L}(\phi) - 2\delta^{(4)}(0)\ln(\phi_\alpha^2). \tag{1.11}$$

(For an $O(N)$-invariant theory we would have had

$$\bar{\mathscr{L}}(\phi) = \mathscr{L}(\phi) - \tfrac{1}{2}N\delta^{(4)}(0)\ln(\phi_\alpha^2) \tag{1.12}$$

where α runs from 1 to N.) Had we worked in units in which $\hbar \neq 1$, the regularized second term in (1.11) vanishes in the classical limit $\hbar \to 0$ and this represents a hidden quantum mechanical effect. Of course, if we were to adopt dimensional regularization, in which $\delta^{(4)}(0) \equiv 0$, we would automatically recover the canonical result. As we indicated earlier, the aim of Klauder and others is to show that, while canonical quantization is never internally inconsistent, it may not always be appropriate. Changes as in (1.6) are most easily understood in lattice regularization. From this viewpoint, we are considering the possibility that, in the continuum limit, the resulting theory will be in a different universality class from the canonical theory. As we said, it was partly in anticipation of this that scale-invariant changes of measure were proposed. In such a formalism dimensional regularization is known to be overrestrictive, since changes of measure along the lines of (1.6) are able to change the equivalence class of the theory (Chen *et al* 1982, Chen and Fisher 1985, Liu and Fisher 1990). [An analytic, rather than numerical, demonstration of the restrictiveness of dimensional regularization is provided by the large-N limit of (1.12) (Rivers 1983 and Gent 1984).]

Let us now return to the approach of de Alfaro *et al*, in which the theory (1.5) is seen as a low-energy model with a phenomenological cut-off Λ (or, equivalently, a lattice length $a = \Lambda^{-1}$). Then $\delta^{(4)}(0)$ is most naturally interpreted as

$$\delta^{(4)}(0) \simeq \Lambda^4 \tag{1.13}$$

to give an 'effective' potential

$$\bar{V}(\phi) = \lambda(\phi_\alpha^2)^2 - 2\Lambda^4 \ln(\phi_\alpha^2) \tag{1.14}$$

from (1.11). The choice (1.13) is the most naive way in which to introduce an energy scale, but for the moment we take it seriously. Insofar as the 'classical' potential $\bar{V}(\phi)$ describes the symmetry-breaking of the model correctly, the infinite 'witches' hat' spike at $\phi_\alpha = 0$ forces the minima to lie on the surface $|\phi_\alpha^2| = \Lambda^2/\sqrt{\lambda} > 0$. A naive saddle-point approximation to (1.10) that identifies the free energy of the system with $\bar{V}(\phi)$ is then compatible with (1.1), with f_π determined from the $\phi_4 \neq 0$ minimum as

$$f_\pi = \Lambda(1/\lambda)^{1/4} \tag{1.15}$$

thus breaking both the SO(4) and conformal invariance.

The over-riding problem, unanswered by de Alfaro *et al* is whether the symmetry-breaking of (1.11), induced by the change in measure, survives radiative corrections (i.e. renormalization). In the next section we shall implement renormalization-group block transformations upon the cut-off theory (1.14) in the manner suggested by Wilson (1972). The calculation is fairly primitive, but it shows that symmetry-breaking can, indeed, be induced by a change in measure, provided the coupling strength λ is not too large. This justification of the approach of de Alfaro *et al* is not an artefact of the simplification that we have made. In the final sections of this paper it will be shown that a symmetry-breaking phase survives a more sophisticated analysis based on the Callan–Symanzik renormalization-group equations.

The middle part of the paper is concerned with whether symmetry-breaking can be induced by a change in measure in the continuum-field theory ($\Lambda \to \infty$), for which overt symmetry-breaking is no longer possible. We have every expectation that this is not the case (Klauder 1981, Ogielski 1983 and Gent *et al* 1986), but a specific calculation needs to be performed. We find that a high-temperature series analysis is sufficient to demonstrate non-vanishing logarithmic terms in the continuum-limit critical behaviour that enforce triviality. Hence the analysis of de Alfaro *et al* is particular to effective theories *with a cut-off.*

2. Symmetry breaking in the cut-off theory

For simplicity, we consider the case of a *single* real scalar field $\phi(x)$ in $d=4$ Euclidean dimensions, quantized with respect to the conformally-invariant measure

$$d\Omega_C(\phi) = \prod_x (d\phi(x)|\phi(x)|). \tag{2.1}$$

This one-field model is complicated enough to display the logarithmic effect of the change of measure on the Lagrangian density

$$\mathscr{L}_E(\phi) = \tfrac{1}{2}(\partial_\mu \phi)^2 + \lambda\phi^4 \tag{2.2}$$

while only possessing the discrete reflection symmetry $\phi \to -\phi$, and hence evading problems of Goldstone modes present in the chiral model.

As we saw earlier, if

$$\bar{V}(\phi) = \lambda\phi^4 - \tfrac{1}{2}\delta^{(4)}(0)\ln\phi^2 = \lambda\phi^4 - \tfrac{1}{2}\Lambda^4 \ln(\phi^2/\Lambda^2) \tag{2.3}$$

is taken to determine the symmetry of the theory, the effect of the $\log\phi^2$ term is to induce symmetry-breaking in which, as a first guess

$$v = |\langle\phi\rangle| = \Lambda(\tfrac{1}{4}\lambda)^{1/4} \tag{2.4}$$

the position of the minimum of $\bar{V}$ for a given cut-off Λ. Because of the absence of m^2 terms, it is convenient to work with the dimensionless field $\tilde{\phi} = \phi/\Lambda$, in terms of which

$$\bar{V}(\tilde{\phi}) = \Lambda^4(\lambda\tilde{\phi}^4 - \tfrac{1}{2}\ln\tilde{\phi}^2) \tag{2.5}$$

with minima at $v = \tilde{\phi} = (\tfrac{1}{4}\lambda)^{1/4}$.

Beyond this semiclassical approximation the position is unclear. The logarithmic divergence at the origin of ϕ is the weakest possible, and it is not obvious whether the symmetry-breaking effects are stable to quantum fluctuation. Conventional perturbative renormalization arguments are of little use here, and an alternative approach is necessary. Such a method has been given by Fukuda (1976), who has readapted Wilson's renormalization group formulae (Wilson 1972) to a theory with a cut-off.

We begin with the definition of the effective potential (free energy) $V(\bar{\phi})$ (Fukuda and Kyriakopoulos 1975) as the $\omega \to \infty$ limit for

$$e^{-\omega V(\bar{\phi})} = \int d\Omega_C(\phi)\, \delta\left(\frac{1}{\omega}\int d^4x\, \phi(x) - \bar{\phi}\right) \exp\left(-\int \mathscr{L}\, d^4x\right) \tag{2.6}$$

where $\omega = \int d^4x$ denotes spacetime volume. This can be expressed in terms of Fourier components $\Phi(k)$ as

$$\begin{aligned} e^{-\omega V(\bar{\phi})} &= \int d\Omega_C(\phi)\, \delta(\Phi(k=0) - \bar{\phi}) \exp\left(-\int \mathscr{L}\, d^4x\right) \\ &= \int d\Omega_C(\phi') \exp\left(\int d^4x\, \mathscr{L}(\bar{\phi} + \phi')\right) \end{aligned} \tag{2.7}$$

where ϕ' is the deviation of ϕ from $\Phi(k=0)$,

$$\phi = \Phi(k=0) + \phi'. \tag{2.8}$$

Formula (2.6) enables us to obtain $V(\bar{\phi})$ by integrating out all $\Phi(k)$ except $\Phi(k=0)$, which we set equal to $\bar{\phi}$. However, in practice it is by no means an easy task. To get a rough idea of the effect of integration over $\Phi(k)$ we use Wilson's approximate recursion formula (Wilson 1972).

In (2.7) the range of momenta $0 < |k| < \Lambda$, bounded by the phenomenological cut-off Λ, is divided in regions

$$2^{-i-1}\Lambda < |k| < 2^{-i}\Lambda \tag{2.9}$$

where $i = 0, 1, 2, \ldots$. First we integrate out $\Phi(k)$, $\frac{1}{2}\Lambda < |k| < \Lambda$, leaving the other $\Phi(k)$ untouched. Taking the logarithm of this result gives an effective Lagrangian density $\mathscr{L}^{(1)}$ which now only contains $\Phi(k)$ with $|k| < \frac{1}{2}\Lambda$. Repeating this process yields a sequence of effective Lagrangian densities

$$\mathscr{L} \to \mathscr{L}^{(1)} \to \mathscr{L}^{(2)} \to \mathscr{L}^{(3)} \to \ldots. \tag{2.10}$$

The limit of this sequence, if it exists, is a function of $\Phi(k=0) = \bar{\phi}$ and is nothing but $V(\bar{\phi})$. Wilson's approximation consists of replacing $e^{i\boldsymbol{k}\cdot x}$ by 1 for small $\boldsymbol{k}$, and by ± 1 for large $\boldsymbol{k}$, when it occurs in the Fourier expansion.

With this approximation we construct a sequence of approximants to the effective potential $V(\bar{\phi})$. We shall not recreate the arguments of Fukuda (1976), but merely quote the results as they apply to the case in hand. Each step in the integration gives rise to an 'effective' potential $U(\bar{\phi})$. If, at the lth step, we denote this potential by $U_l(\bar{\phi})$, then $U_{l+1}(\bar{\phi})$ is given in terms of $U_l(\bar{\phi})$ by

$$U_{l+1}(\bar{\phi}) = -2^{-4l} \ln\{f(U_l(\bar{\phi}))/f(U_l(0))\} \tag{2.11}$$

where

$$f(U_l(\bar{\phi})) = \int_{\infty}^{\infty} dy\, \exp[-2^{2l}y^2 - 2^{4l-1}(U_l(y+\bar{\phi}) + U_l(-y+\bar{\phi}))] \tag{2.12}$$

in which d is taken to be four again.

Beginning with

$$U_0(\bar{\phi}) = \lambda\bar{\phi}^4 - \frac{1}{2}\ln \bar{\phi}^2 \tag{2.13}$$

the recurrence relation generates a sequence of U_l's. The limit

$$\lim_{l \to \infty} U_l(\bar{\phi}) = V(\bar{\phi}) \tag{2.14}$$

is the effective potential we need. We note that $V(\bar{\phi})$ is *independent* of the cut-off Λ, except through $\bar{\phi}=\phi/\Lambda$. Further, by construction $V(\bar{\phi})$ is *convex*, as befits a free energy, irrespective of the central spike in U.

The results of the numerical integrations are given in figures 1 and 2. We note that for λ small (figure 1), e.g. $\lambda=0.25$, convergence starts after a few iterations (in fact seven iterations are sufficient for this value of λ). $V(\bar{\phi})$ becomes flat, assuming the standard 'bucket' shape. In particular, $V(\bar{\phi})$ is flat for $|\bar{\phi}|<\bar{\phi}_C$, where $\bar{\phi}_C$ approximates

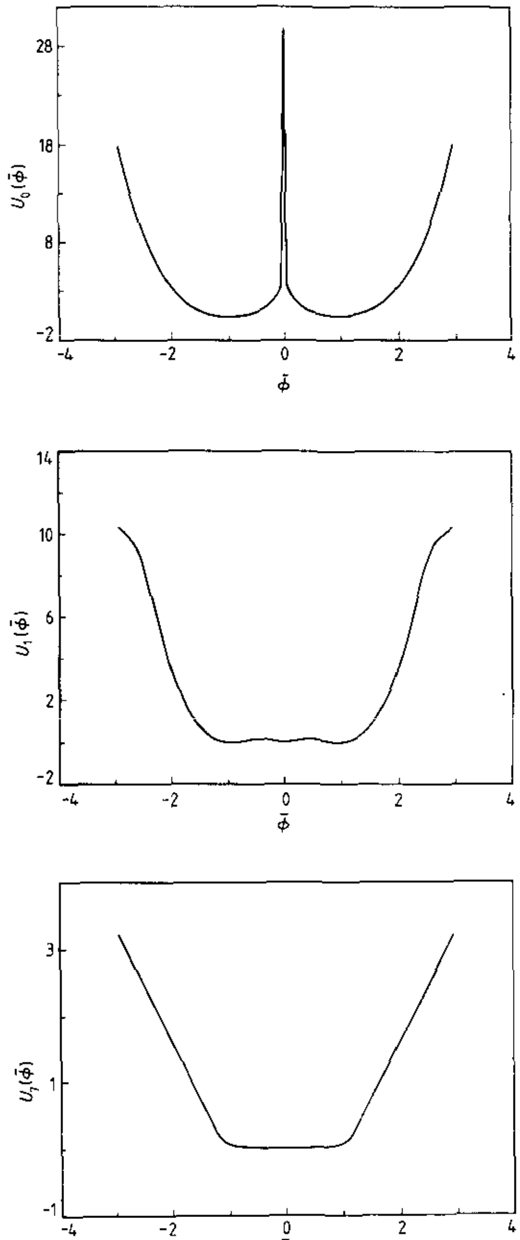

Figure 1. Plots of $U_0(\bar{\phi})$, $U_1(\bar{\phi})$ and $U_7(\bar{\phi})$ as a function of $\bar{\phi}$ at $\lambda=0.25$.

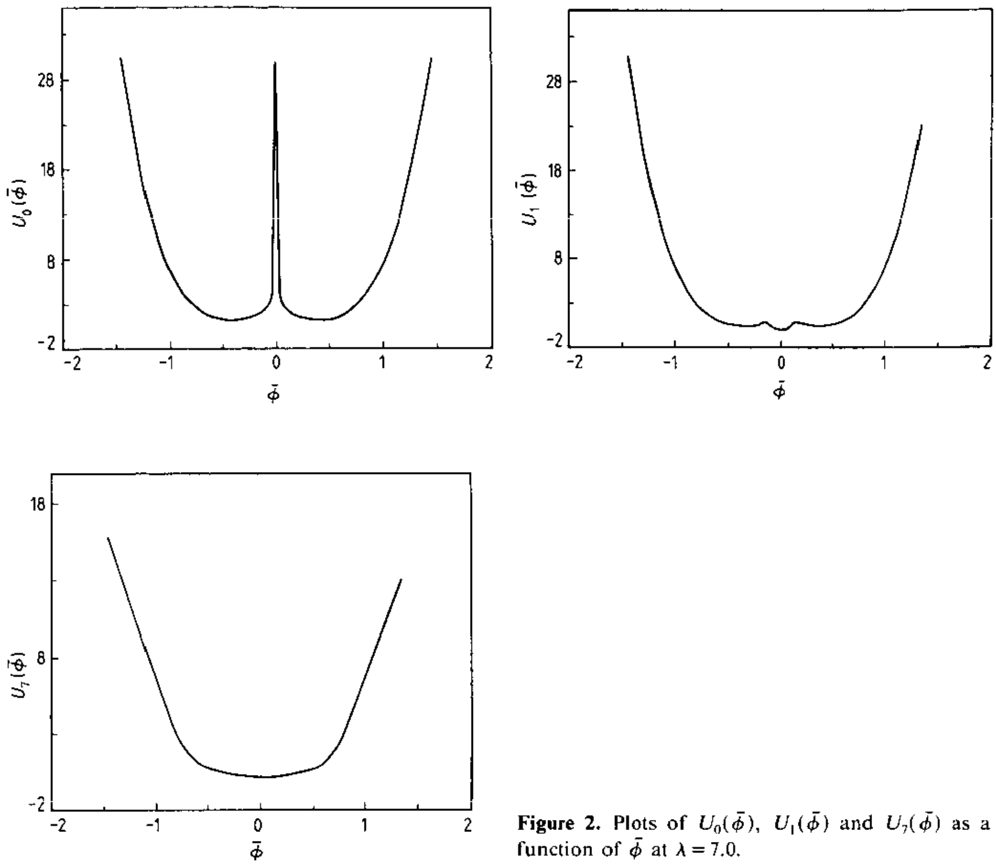

Figure 2. Plots of $U_0(\bar{\phi})$, $U_1(\bar{\phi})$ and $U_7(\bar{\phi})$ as a function of $\bar{\phi}$ at $\lambda = 7.0$.

the classical value

$$\bar{\phi}_C \simeq (\tfrac{1}{4}\lambda)^{1/4} \tag{2.15}$$

with no apparent shift during iterations. This supports the scenario of de Alfaro *et al.*

As λ becomes large (figure 2) there comes a critical λ_C ($\lambda_C \leqslant 7.0$) above which the $\lambda\phi^4$ behaviour completely dominates. Thus no symmetry-breaking is seen after renormalization. In other words, the theory possesses a symmetric phase and a broken-symmetric phase separated by a critical value $\lambda = \lambda_C$. Spontaneous symmetry-breaking occurs only when $\lambda < \lambda_C$. When, later, we reconsider the theory from its high-temperature expansion, we shall estimate λ_C directly.

Had we not taken the conformally-invariant choice, but the more general scale-covariant measure (Klauder 1981, Ogielski 1983 and Gent *et al* 1986)

$$\mathrm{d}\tilde{\Omega}(\phi) = \mathrm{d}\Omega(\phi) \prod_x |\phi(x)|^{-B} \tag{2.16}$$

also compatible with (1.15), the results are qualitatively unchanged for $B < 0$. For example, on repeating the above analysis for $B = -2.0$ we find a critical coupling $\lambda_C \simeq 15$.

For $B > 0$ the central logarithmic spike in $\bar{V}(\phi)$ becomes a logarithmic dip, and we only find a symmetric phase, just as in the semi-classical analysis.

3. Removing the cut-off

If we take $\Lambda\to\infty$ directly in (2.3) the semi-classical potential $\tilde{V}$ diverges. De Alfaro *et al* restricted themselves to finite Λ, and were able to avoid considering the continuum case. However, we know that this divergence is an artefact of the semi-classical approximation and that, if properly normalized, the limit will most likely exist. However, there is no guarantee that it displays symmetry-breaking, or even that it describes an interacting theory. Therefore, it is worth trying to understand better the effect of the conformal measure invoked by de Alfaro *et al* in the infinite cut-off limit. As we said earlier, non-canonical measures have been proposed as a step towards more fundamental (e.g. non-trivial, renormalizable) theories and the implications of these measures have not been fully understood.

To see what the renormalized $\Lambda\to\infty$ conformal scalar theory is, we consider its generating functional

$$Z[j]=\int \mathrm{d}\Omega_{\mathrm{C}}(\phi)\exp\left[-\int \mathrm{d}^d x(\mathscr{L}(\phi)-j\phi)\right] \tag{3.1}$$

where, although we are working in $d=4$ dimensions, we have left d explicit. The Lagrangian density $\mathscr{L}(\phi)$ is given by (1.4). The most convenient tactic is to put the theory on a hypercubic lattice with lattice spacing a. $Z[j]$ can be written in terms of dimensionless spin variables

$$\sigma_k=a^{(d-2)/2}\phi(ak)/\sqrt{K} \tag{3.2}$$

situated at sites ak, $k=(k_1,k_2,k_3,k_4)\in\mathbb{Z}^4$. The quantity K is an *inverse temperature* (in units in which Boltzmann's constant is unity). In terms of σ, a, K, $Z[j]$ becomes

$$Z[J]=\int\prod_k(\mathrm{d}\sigma_k|\sigma_k|)\exp\left\{-A\sum_k\sigma_k^2-u\sum_k\sigma_k^4+\sum_k J_k\sigma_k+\tfrac{1}{2}\sum_{lm}\sigma_l K_{lm}\sigma_m\right\} \tag{3.3}$$

where

$$A=\mathrm{d}K \tag{3.4}$$

$$u=K^2\lambda a^{4-d} \tag{3.5}$$

$$J_{\mathrm{K}}=\sqrt{K}\,a^{(2+d)/2}j(ak). \tag{3.6}$$

The spin–spin coupling term K_{lm} (with no diagonal component) is

$$K_{lm}=\begin{cases}K & l,\ m \text{ nearest neighbours}\\ 0 & \text{otherwise.}\end{cases} \tag{3.7}$$

Prior to taking the continuum limit, $a^{-1}=\Lambda$ provides a natural momentum cut-off. If we were to put the change in measure into the exponent it would exactly correspond to making the identification (2.3).

On defining the single-site spin distribution $\mu(\sigma)$ by

$$\mathrm{d}\mu(\sigma)=\mathrm{d}\sigma|\sigma|\exp[-(A\sigma^2+u\sigma^4)] \tag{3.8}$$

$Z[J]$ can finally be written as

$$Z[J]=\int\prod_k \mathrm{d}\mu(\sigma_k)\exp\left(\tfrac{1}{2}\sum_{lm}\sigma_l K_{lm}\sigma_m-\sum_k\sigma_k J_k\right) \tag{3.9}$$

a continuous-spin ferromagnetic of the type discussed in great detail by Baker and Kincaid (1981) and others. The change of variables in (3.4) onwards maps λ into u, K. We eliminate the extra parameter by imposing the Baker–Kincaid normalization condition

$$\int \mathrm{d}\mu(\sigma)\, \sigma^2 = \int \mathrm{d}\mu(\sigma). \tag{3.10}$$

That is,

$$\int \mathrm{d}\sigma|\sigma|\, \sigma^2 \exp(-A\sigma^2 - u\sigma^4) = \int \mathrm{d}\sigma|\sigma| \exp(-A\sigma^2 - u\sigma^4). \tag{3.11}$$

This fixes $A = A(u)$ as a calculable function of u, or vice versa. Because of the factor $|\sigma|$ in $\mathrm{d}\mu(\sigma)$, that depresses small σ, (3.9) describes an Ising-like system.

To recover the continuum limit we need to go to the critical region of the system where, for a second-order phase transition, the correlation length becomes infinite in lattice units. Since correlation length has the interpretation of inverse mass for the quanta of the theory (and is fixed in cm), going to the critical region corresponds to driving the lattice size $a \to 0$.

The spin correlation functions of interest to us are

(i) the susceptibility

$$\chi = \sum_k \langle \sigma_0 \sigma_k \rangle_C \tag{3.12}$$

(ii) the fourth cumulant

$$\chi_{(2)} = \sum_{klm} \langle \sigma_0 \sigma_k \sigma_l \sigma_m \rangle_C \tag{3.13}$$

and

(iii) the second moment of the spin–spin correlation function

$$\mu_{(2)} = \sum_k \frac{k_\mu k_\mu}{a^2} \langle \sigma_0 \sigma_k \rangle_C \tag{3.14}$$

where spin averages are defined by

$$\langle F\{\sigma_k\}\rangle = Z[0]^{-1} \int \prod_k \mathrm{d}\mu(\sigma_k) F\{\sigma_k\} \exp(\tfrac{1}{2}\Sigma\sigma K\sigma) \tag{3.15}$$

and c denotes the connected part.

In terms of the above we can define the correlation length

$$\xi = \left(\frac{\mu_{(2)}}{2\,\mathrm{d}\chi}\right)^{1/2} \tag{3.16}$$

and the dimensionless scale-invariant renormalized coupling constant

$$g_{\mathrm{R}} = -\frac{\chi_{(2)}}{\xi^d \chi^2}. \tag{3.17}$$

Assume for the moment that the system displays a second-order phase transition as the temperature approaches a critical value $K \to K_C$. The critical behaviour of the correlation functions is taken to be

$$\chi = f_1(K)(K_C - K)^{-\gamma} \tag{3.18a}$$

$$\chi_{(2)} = f_2(K)(K_C - K)^{-\gamma-2\Delta} \tag{3.18b}$$

$$\mu_{(2)} = f_3(K)(K_C - K)^{-\gamma-2\nu} \tag{3.18c}$$

$$\xi = f_4(K)(K_C - K)^{-\nu} \tag{3.18d}$$

whence

$$g_R = f_5(K)(K_C - K)^{\kappa} \tag{3.18e}$$

where

$$\kappa = d\nu + \gamma - 2\Delta \tag{3.19}$$

and f_i ($i = 1, \ldots, 5$) are analytic in K (neglecting confluent and logarithmic singularities at present). Although $d = 4$ we have continued to display it explicitly. In the scaling (continuum) limit $K \to K_C$ the magnitude of κ is crucial, as it determines whether the theory is *trivial* ($\kappa > 0$) or *non-trivial* ($\kappa = 0$).

As a guide to the value of κ (and K_C), and to make a *second-order* transition plausible, we conclude this section by evaluating the Landau mean-field approximation to (3.7) (see, for instance, Brézin *et al* 1976). We would expect this to be correct for $d > 4$, and a good guide for $d = 4$. Inserting the identity

$$\int \prod_k \mathrm{d}\chi_k \exp\left(-\tfrac{1}{2}\sum_{i\neq j} \chi_i K_{ij}^{-1}\chi_j + \sum \sigma_i\chi_i\right) = \text{constant} \times \exp\left(\tfrac{1}{2}\sum_{i\neq j} \sigma_i K_{ij}\sigma_j\right) \tag{3.20}$$

into (3.7) gives (up to normalization)

$$Z[J] = \int \prod_k \mathrm{d}\chi_k \exp\left\{-\frac{1}{2c}\sum_{i\neq j}(\chi_i - J_i)K_{ij}^{-1}(\chi_j - J_j) + \sum_k A(\chi_k)\right\}\bigg|_{C=1} \tag{3.21}$$

where

$$A(\chi) = \ln \int \mathrm{d}\mu(\sigma)\, \mathrm{e}^{\sigma\chi}. \tag{3.22}$$

Expanding in powers of c gives, to leading order, the saddle-point result that $W[J] = \ln Z[J]$ is given by

$$W_{\mathrm{MFA}}[J] = -\tfrac{1}{2}\sum_{i\neq j}(\bar\chi_i - J_i)K_{ij}^{-1}(\bar\chi_j - J_j) + \sum_k A(\bar\chi_k) \tag{3.23}$$

where

$$K_{ij}^{-1}(\bar\chi_j - J_j) = A'(\bar\chi_i). \tag{3.24}$$

In terms of the magnetization $M_i = A'(\bar\chi_i)$ the effective action is given as the usual Legendre transform

$$\Gamma_{\mathrm{MFA}}[M] = -W_{\mathrm{MFA}}[J] + \sum_k J_k M_k = -\tfrac{1}{2}\sum_{i\neq j} M_i K_{ij} M_j + \sum_j B(M_j). \tag{3.25}$$

For *constant magnetization* $M_j = M$ (and constant $\bar\chi_i = \bar\chi$)

$$B(M) = -A(\bar\chi) + M\bar\chi \tag{3.26}$$

permits a Taylor expansion

$$B(M)=\frac{\alpha}{2!}M^2+\frac{\beta}{4!}M^4+\frac{\rho}{6!}M^6+\mathrm{O}(M^8) \tag{3.27}$$

in which ($I_2=1$ from (3.10))

$$\alpha=\frac{1}{I_2}=1 \tag{3.28}$$

$$\beta=-\frac{1}{I_2^4}(I_4-3I_2^2)=-(I_4-3) \tag{3.29}$$

and

$$\rho=10(I_4-3)^2-(I_6-15(I_4-2)) \tag{3.30}$$

where

$$I_{2n}=\int \mathrm{d}\mu(\sigma)\,\sigma^{2n}\left(\int \mathrm{d}\mu(\sigma)\right)^{-1} \tag{3.31}$$

are the normalized moments of $\mu(\sigma)$.

On an N-site lattice with constant magnetization the resultant free energy density $\Gamma_{\mathrm{MFA}}[M]/N$ then takes the form

$$\Gamma_{\mathrm{MFA}}[M]/N=\tfrac{1}{2}(1-2\,\mathrm{d}K)M^2+\frac{\beta}{4!}M^4+\frac{\rho}{6!}M^6+\mathrm{O}(M^8) \tag{3.32}$$

displaying a phase transition at the critical temperature

$$T_{\mathrm{C}}^{\mathrm{MF}}=(K_{\mathrm{C}}^{\mathrm{MF}})^{-1}=2d. \tag{3.33}$$

At $K=K_{\mathrm{C}}$, A and u take the critical values

$$\begin{aligned} A_{\mathrm{C}}^{\mathrm{MF}}&=\mathrm{d}K_{\mathrm{C}}^{\mathrm{MF}}=\tfrac{1}{2}\\ u_{\mathrm{C}}^{\mathrm{MF}}&=0.145\,97 \end{aligned} \tag{3.34}$$

(the latter following from (3.11)). By the recurrence relation for the I_{2n},

$$I_{2n+4}=\frac{-A}{2u}I_{2n+2}+\frac{n+1}{2u}I_n \tag{3.35}$$

for $A=A_{\mathrm{C}}$ and $u=u_{\mathrm{C}}$ we have

$$\rho\simeq 8>0$$

and

$$\beta\simeq 1.25>0 \tag{3.36}$$

signalling a *second-order* transition.

Furthermore, in $d=4$ dimensions, equation (3.5) gives a critical value for λ,

$$\lambda_{\mathrm{C}}^{\mathrm{MF}}=u_{\mathrm{C}}^{\mathrm{MF}}/(K_{\mathrm{C}}^{\mathrm{MF}})^2=64u_{\mathrm{C}}^{\mathrm{MF}}\simeq 9.342. \tag{3.37}$$

Whereas the cut-off theory permitted any value of λ, the continuum limit drives λ to λ_{C}, the critical value of the cut-off theory. The mean-field result is fairly close to the value $\lambda_{\mathrm{C}}\sim 7$ of section 2. At a later stage we shall use u_{C} of (3.34) as a starting-point for a more sophisticated calculation. A little more work gives values for the critical exponents

$$\nu_{\mathrm{MF}}=\tfrac{1}{2}$$
$$\gamma_{\mathrm{MF}}=1 \qquad (3.38)$$
$$\Delta_{\mathrm{MF}}=\tfrac{3}{2}$$

whence (for $d=4$)

$$\kappa_{\mathrm{MF}}=0. \qquad (3.39)$$

These lattice results are identical to those for the *canonically quantized* $\lambda\phi^4$ theory in the same mean-field approximation. In this case we know that $\kappa=0$ does *not* imply a non-trivial theory, although from our previous comments it should.

The reason is that the forms presented in (3.18) are too simple. In $d=4$ dimensions logarithms are generally present, giving correlation functions with the behaviour

$$F\sim(K_{\mathrm{C}}-K)^a|\ln(K_{\mathrm{C}}-K)|^b \qquad (3.40)$$

as $K\to K_{\mathrm{C}}$. Thus, even if the critical powers are those of mean field theory, the residual logarithms can still enforce triviality upon the theory.

4. The high-temperature expansion

Baker and Kincaid (1981) have calculated the high-temperature series up to $N=10$th order in K (recently Lüscher and Weisz (1988b) have extended the series up to $N=14$th order in K) for χ, $\chi_{(2)}$ and $\mu_{(2)}$ in the form

$$F(K)=\sum_{n=0}^{N} f_nK^n \qquad (4.1)$$

by directly expanding the exponential in (3.12). The coefficients f_n are given in terms of the moments I_{2n} of (3.31).

For the moment, let us forget the likely existence of logarithms as indicated in (3.40) and pretend that only powers are present. If the true behaviour of the correlation function $F(K)$ is

$$F(K)\sim f(K)(K_{\mathrm{C}}-K)^{\varepsilon} \qquad (4.2)$$

as $K\to K_{\mathrm{C}}$, how do we identify ε from a partial series like (4.1)?

The most convenient method is to use Padé approximants for

$$\frac{\mathrm{d}}{\mathrm{d}K}\ln F(K)=\frac{\varepsilon}{K_{\mathrm{C}}-K}+\frac{\mathrm{d}}{\mathrm{d}K}\ln f(K) \qquad (4.3)$$

which is represented by the (shorter) series

$$\frac{\mathrm{d}}{\mathrm{d}K}\ln F(K)=\sum_{m=0}^{N-1} g_mK^m \qquad (4.4)$$

in which the g_m are determined directly from the f_n after identifying the logarithm by its Taylor series. Padé approximants are ideally suited for estimating pole positions and residues and this method for calculating critical indices and temperatures is termed the 'd-log Padé' method.

Recall from (5.3.4) that

$$A = \mathrm{d}K \tag{4.5}$$

and the normalization condition (3.10)

$$I_2(A, u) = 1. \tag{4.6}$$

If we fix u, then A is also fixed through (4.6). As in the case of the mean-field theory of the previous section, the system is determined uniquely at its critical temperature K_C. In practice, K_C (and hence A_C through (4.5)) is also determined from the d-log Padé series (4.4). Consistency thus requires that u is driven to its critical value u_C, at which value $A(u_C) = A_C$.

It is impossible to pin down u_C exactly numerically. To obtain a preliminary estimate we begin with the mean-field result of (3.34), $u_C^{MF} = 0.145\,97$. From this first guess we use the d-log Padé approximants to calculate $K_C^{(i)}$ for a few different values of $u^{(i)}$ around u_C^{MF}, and calculate the corresponding $A_C^{(i)}$ via (4.6). On plotting $A_C^{(i)}/K_C^{(i)}$ against $u_C^{(i)}$, we can track down a point u_C at which $A_C^{(i)}/K_C^{(i)} = d \simeq 4.00$, using linear interpolation (figure 3). We then repeat the process sweeping a smaller region around this value of u^* to obtain a better value.

In our case, we have (up to variation between different approximants)

$$u_C = 0.1254 \tag{4.7a}$$

$$A_C = 0.5643 \tag{4.7b}$$

$$K_C = 0.140\,81\ (0.000\,07) \tag{4.7c}$$

$$d = A_C/K_C = 4.007\,52\ (0.002\,3). \tag{4.7d}$$

This gives a critical value of λ

$$\lambda_C = u_C/K_C^2 = 6.325 \tag{4.8}$$

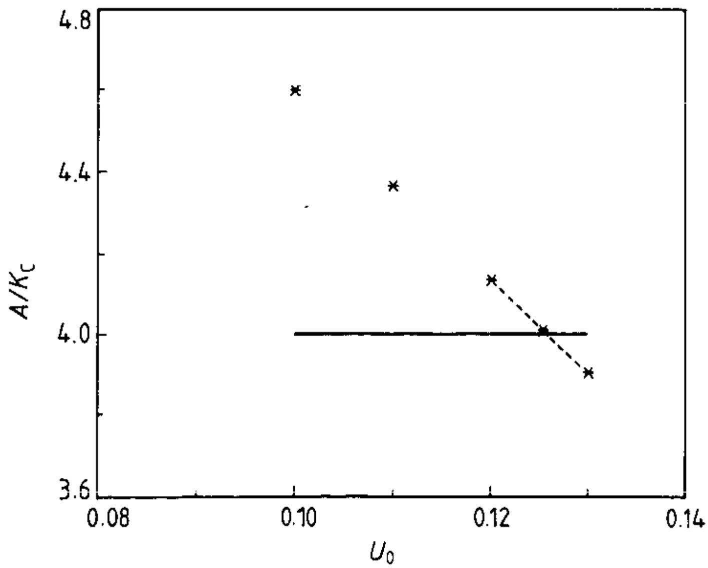

Figure 3. Plot of A/K_C as a function of U_0. The full line dentoes $A/K_C = 4$ and the broken line indicates the extrapolated values of A/K_C within the range $U_0 \in (0.12, 0.13)$.

lower than the mean-field value (3.37), and compatible with the numerical analysis of section 2.

(If, alternatively, we had chosen the normalization condition $A = 4.0$, for which $K_C = 1$, then a repeat of the high-temperature Padé analysis gives $\lambda_C = u_C = 6.327$. This confirms that λ_C is independent of how the normalization condition is fixed.)

What matters is that, at the value (4.4) of u_C, we find that

$$\gamma = 1.091\,26\ (0.001\,73) \tag{4.9}$$

which deviates significantly from the mean-field value $\gamma = 1$, whereby

$$\kappa = 0.314\ (0.05). \tag{4.10}$$

This value is also significantly different from the mean-field value of zero (3.39), and is similar to the canonical results Baker and Kincaid (1981). At this level of analysis we would expect the theory of (3.3) to have a *trivial* continuum limit.

Because of the numerical similarity to the canonical theory which, as we have said, had logarithmic corrections we should consider such corrections in this case. Suppose again that the scaling behaviour of the susceptibility becomes

$$\chi = f_2(K)(K_C - K)^{-\gamma}|\ln(K_C - K)|^{-Z\gamma} \tag{4.11}$$

where f_2 is analytic at $K = K_C$ and $\gamma = 1$, the mean-field exponent.

We can use the high-temperature series again to estimate z, following Adler and Privman (1981, 1982). The method consists of using the χ series to construct the series for the function

$$g(K) = \frac{1}{\gamma}(K_C - K)\ln(K_C - K)\left(\frac{\chi'(K)}{\chi(K)} + \frac{\gamma}{K - K_C}\right) \tag{4.12}$$

which has the property

$$\lim_{K \to K_C} g(K) = z. \tag{4.13}$$

Thus the standard d-log Padé approximants to $g(K)$ evaluated at K_C will provide us with the estimate for z. We know the value of γ, yet the value of K_C in (4.11) remains unknown. To overcome this problem we use the K_C value $K_C = 0.1408$ of the previous analysis (equation (4.7c)) as a preliminary estimate. Then we evaluate different order Padé approximants to z for different values of K_C in the neighbourhood of the initial choice. These approximants give rise to a family of $Z(K_C)$ curves on the $Z - K_C$ plane, which are expected to converge near the true value of Z and K_C.

The area of convergence is located by the standard, yet somewhat subjective 'windowing' approach of Adler *et al* (1981, 1982) (see also Vladikas *et al* (1987)) in figure 4. From this z is found to be

$$z = -0.29\ (0.03). \tag{4.14}$$

The magnitude of z suggests that the presence of logarithmic corrections to mean-field scaling is genuine and is very close to the *canonical* $\lambda\phi^4$ renormalization group prediction $z = -\frac{1}{3}$.

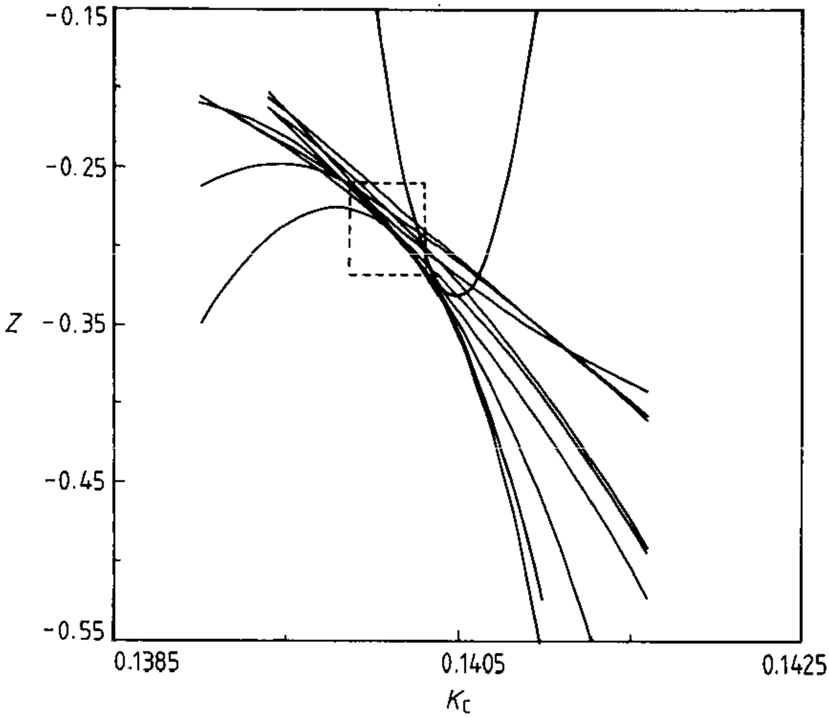

Figure 4. Plot of the Padé approximants to the logarithmic correction factor Z as a function of K_C at $U_0 = 0.1254$. The 'window' isolates the area of convergence.

In any case, our results (4.10) and (4.14) suggest, in their different ways, that the conformal scalar theory is as trivial as its canonical counterpart.

5. Renormalization group analysis

Given the crudity of the Wilson block-renormalization approximation in section 2, we need to dispel thoughts that our conclusions given there are dependent on the details of the approximation. Let us return to the equivalent spin-system (3.8). Rather than continue with the tactics of Baker and Kincaid (1981) we shall adopt the formalism of Lüscher and Weisz (1987, 1988a). To facilitate comparison between our work and that of Lüscher and Weisz (1987, 1988a), it is convenient to change the formalism slightly. Instead of imposing the constraint (3.10), it will be convenient to fix

$$K = 1 \tag{5.1}$$

throughout. From this viewpoint there is a critical coupling $\lambda = \lambda_C$ (for which $K_C(\lambda_C) = 1$) rather than a critical temperature. The value of λ_C is taken from (4.8). For $\lambda > \lambda_C$ we are in the symmetric phase, and for $\lambda < \lambda_C$ in the symmetry-broken phase.

From our previous discussion we believe the continuum limit $\lambda = \lambda_C$ to be *trivial*, in $d = 4$ dimensions. That is, the critical behaviour of the renormalized coupling constant g_R is such that it *vanishes* there. However, this triviality need not make the theory useless for the description of the particles that are the underlying quanta, provided we allow for a large but finite ultraviolet cut-off Λ which we continue to identify with a^{-1} in the lattice theory. If, for low-energy processes where E^2/Λ^2 is small (here E is a typical energy in the process), the corresponding amplitudes are universal, up to terms $O(E^2/\Lambda^2)$, we have a sensible effective theory.

Because our starting theory (1.4) has vanishing bare mass the vacuum expectation value $v_R = \langle\phi\rangle$, renormalized mass m_R and Λ will be mutually constrained. This is as we would have expected from semi-classical analysis of (2.3), for which v is given by (2.4). The classical particle mass m induced by the symmetry breaking satisfies

$$m^2 = 8\sqrt{\lambda}\,\Lambda^2 = 4\Lambda^4/v^2 \tag{5.2}$$

whence

$$vm = 2\Lambda^2. \tag{5.3}$$

Unfortunately, from (5.3) it is apparent that we cannot simultaneously impose $\Lambda/m \gg 1$, $\Lambda/v \gg 1$, as we would like, for the classical cut-off theory to make sense. (This point was glossed over by de Alfaro *et al.*) However, the renormalized theory is more subtle and we shall see that (5.3) is over-restrictive.

We have already shown how the high-temperature series for (3.3) (i.e. the expansion in K at fixed A, u) can be used in the vicinity of the continuum limit in the symmetric phase. To demonstrate the circumstances under which the non-scaling terms $O(E^2/\Lambda^2)$ can be ignored we adopt the tactics of Lüscher and Weisz (1987, 1988a), in which the high-temperature series results are used as boundary values for the renormalization group (RG) equations.

There is a problem. The non-canonical nature of the single-site distribution (3.8) has no effect on our ability to compute the high-temperature series, which only uses the moments of $\mu(\sigma)$. However, the RG equations require a series expansion in the renormalized coupling g_R for small g_R. Since (3.9) does not permit an expansion about a Gaussian (i.e. Feynman diagrams) we do not have a simple way to proceed.

A solution to this was indicated in section 3, where we found it convenient to rewrite the spin system (3.3) in terms of a *canonical* mean field χ, with translationally-invariant measure $D\chi$. In section 3 we concluded that the continuum conformally-quantized theory was in the same universality class as its canonical counterpart. We shall take this further, in arguing that, near the critical point, it is very plausible that $Z[J]$ can be written *as the canonical spin system* (obtained by rescaling the χ's)

$$Z[\tilde{h}] = \int \prod_k \mathrm{d}S_k \exp\left(\tfrac{1}{2}\sum_{lm} S_l \bar{I}_{lm} S_m - \sum_k (\bar{A}S_k^2 + \bar{u}S_k^4) + \sum_k \tilde{h}_k S_k\right) \tag{5.4}$$

where $\bar{I}_{lm} = \bar{K}(\lambda)$ for nearest neighbours, zero otherwise. Necessarily, the S-system (5.4) is equally trivial in the continuum limit.

We are then in a position to apply the practices of Lüscher and Weisz (1987, 1988a) to determine the behaviour of the cut-off theory. In section 7 we shall set up the correspondence between the σ- and S-systems and determine $\bar{K}(\lambda)$ for $\lambda > \lambda_C$ near the critical point. This enables us to determine the parameters at the critical point sufficiently accurately that we can solve the RG equations in the symmetry-broken phase and, establish the counterpart to the classical result (5.3). In this way we shall quantify the qualitative proposal of de Alfaro *et al* that symmetry breaking can be sensibly adduced to a change of measure in the cut-off theory.

6. The canonical mean field theory

Let us rewrite (3.21) as

$$Z[J]=N\int\prod_k \mathrm{d}\chi_k \exp\Big(-\tfrac{1}{2}\sum \chi_i I_{ij}^{-1}\chi_j+\sum_k A(\chi_k+J_k)\Big) \tag{6.1}$$

where $A(\chi)$ is more conveniently expanded as (cf (3.27))

$$A(\chi)=A(0)+\frac{1}{2!}I_2\chi^2+\frac{1}{4!}(I_4-3I_2^2)\chi^4+\sum_{n=3}^{\infty}\frac{M_{2n}(0)}{(2n)!}\chi^{2n}. \tag{6.2}$$

(It is not necessary to re-express M_{2n} in terms of the I_{2m} since we shall have no use for the details. Remember that, on fixing $K=1$, I_2 is no longer equal to unity.)

To reshape $Z[J]$ in the form (5.4) requires three separate steps. Firstly, we saw in section 3 that the mean field theory provided a fair estimate of λ_C. To obtain this result from (6.1) we Fourier transform I_{ij} as

$$\bar{V}(q)=\sum_j I_{ij}\exp(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_{ij}) \tag{6.3}$$

where $\boldsymbol{r}_{ij}$ denotes the position of all nearest neighbours to the point i. For small $\boldsymbol{q}$, $|\boldsymbol{q}|\ll a^{-1}$, in d dimensions

$$V(q)\simeq 2d\left(1-\frac{a^2}{2d}\boldsymbol{q}^2\right). \tag{6.4}$$

Thus the quadratic term in the exponent of (6.1) becomes, in Fourier transform variables (small $\boldsymbol{q}^2$)

$$\mathscr{L}_0=\tfrac{1}{2}\sum_q (V(\boldsymbol{q})^{-1}-I_2)|\chi(\boldsymbol{q})|^2=\frac{1}{2}\frac{a^2}{2d}\sum_q\left(\frac{\boldsymbol{q}^2}{2d}+\frac{(1-2dI_2)}{a^2}\right)|\chi(\boldsymbol{q})|^2. \tag{6.5}$$

This corresponds to an effective free propagator (or two-spin correlation function)

$$S_0=\frac{2d}{a^2\boldsymbol{q}^2/2d+(1-2dI_2)} \tag{6.6}$$

S_0 shows infrared critical behaviour when

$$\mu_0^2=(1-2dI_2(\lambda))/a^2 \tag{6.7}$$

vanishes. This occurs at $\lambda=\lambda_C$ of (3.37).

Define $\bar{\chi}_k$ by

$$\bar{\chi}_k=2d\chi_k. \tag{6.8}$$

It is sufficient, for the purposes of describing the model near its critical behaviour, to consider only its long wavelength contributions. In this approximation the quadratic contribution to the action

$$\mathscr{L}_0=\tfrac{1}{2}a^2\sum_q(2d\mu_0^2+\boldsymbol{q}^2)|\bar{\chi}(\boldsymbol{q})|^2 \tag{6.9}$$

is again expressible in terms of I_{ij}, rather than I_{ij}^{-1}, to take (6.1) into the form

$$Z[h]=N\int\prod_i \mathrm{d}\bar{x}_i \exp\Big(-\mathscr{L}[\bar{\chi}]+\sum_i[\bar{\chi}_i h_i+\mathrm{O}(h_i^2)]+\mathrm{O}(h_i\bar{\chi}_i^3)\Big) \tag{6.10}$$

where

$$h_i = 2dI_2(\lambda)J_i \tag{6.11}$$

and

$$\mathscr{L}[\bar\chi] = -\tfrac{1}{2}\sum_{i,j} I_{ij}\bar\chi_i\bar\chi_j + \sum_k [d + d(1-2dI_2(\lambda))]\bar\chi_k^2 - \frac{(2d)^4}{4!}\sum_k (I_4 - 3I_2^2)\bar\chi_k^4 - \sum_k \sum_{n=3}^{\infty} \frac{M_{2n}(0)}{(2n)!}(\bar\chi 2d)^{2n}. \tag{6.12}$$

The next step is the observation that, in $d=4$ dimensions, the infrared singular behaviour comes from terms $O(\bar\chi^4)$. We adopt the general belief that the 'irrelevant' terms $O(\bar\chi^6)$ can be taken into account near to the critical point by renormalization of the quadratic and quartic terms. The effect is to replace $\mathscr{L}[\bar\chi]$ by

$$\mathscr{L}[\bar\chi] = -\tfrac{1}{2}\sum_{i,j} a(\lambda)\bar\chi_i\bar\chi_j I_{ij} + \sum_k (a(\lambda)d + b(\lambda))\bar\chi_k^2 + \sum_k g(\lambda)\frac{M_4(\lambda)}{4!}(2d)^4\bar\chi_k^4 \tag{6.13}$$

where

$$M_4(\lambda) = 3I_2^2 - I_4. \tag{6.14}$$

The factors $a(\lambda)$ and $g(\lambda)$ are multiplicative renormalizations of the kinetic term and quartic term respectively, and $b(\lambda)$ summarizes the net result of renormalization for the quadratic coupling. All these are collective effects due to the $O(\bar\chi^6)$ terms whose coefficients are functions of $I_{2n}(\lambda)$, thus determined by λ only.

To simulate the form (3.3) we first consider the quadratic coupling

$$u(\lambda) = \frac{g(\lambda)M_4(\lambda)}{4!}(2d)^4. \tag{6.15}$$

As it stands $u(\lambda)$ is not only λ dependent, but contains an unknown factor. We fix this by rescaling the field $\bar\chi_k$ yet again to S_k,

$$\bar\chi_k = \left[\frac{M_4(\lambda_C)}{M_4(\lambda)g(\lambda)}\right]^{1/4} S_k. \tag{6.16}$$

The Lagrangian now becomes

$$\mathscr{L}[S] = -\tfrac{1}{2}\sum_{i,j}\left(\frac{M_4(\lambda_C)}{M_4(\lambda)g(\lambda)}\right)^{1/2} a(\lambda)S_iS_jI_{ij} + \sum_k (a(\lambda)d + b(\lambda))\left(\frac{M_4(\lambda_C)}{M_4(\lambda)g(\lambda)}\right)^{1/2} S_k^2 + \bar u\sum_k S_k^4 \tag{6.17}$$

where $\bar u = M_4(\lambda_C)/4!(2d)^4$ is known.

The source coupled to S_k is

$$\hat h_k = 2dI_2(\lambda)\left(\frac{M_4(\lambda_C)}{M_4(\lambda)g(\lambda)}\right)^{1/4} J_k \tag{6.18}$$

having forced $\bar u$ to be independent of λ.

The next step is to fix the coefficient of S_k^2 to be λ independent also. This is not possible in (6.17) as it stands. In order to effect this, we extend the long wavelength approximation by modifying the short-range modes when the long-range modes are large and dominate. If we *scale* $\boldsymbol{q}^2$ in (6.5) by a factor $t(\lambda)$ (thus preserving the infrared behaviour) $\mathscr{L}[S]$ of (6.17) becomes

$$\mathscr{L}[S] = -\tfrac{1}{2}\sum_{i,j} \bar{K}(\lambda) S_i S_j I_{ij} + \sum_i \bar{A}(\lambda) S_i^2 + \bar{u}\sum_i S_i^4 \tag{6.19}$$

where

$$\bar{K}(\lambda) = a(\lambda)\left(\frac{M_4(\lambda_C)}{M_4(\lambda)g(\lambda)}\right)^{1/2} t(\lambda) \tag{6.20}$$

$$\bar{A}(\lambda) = (\mathrm{d}a(\lambda)t(\lambda) + b(\lambda))\left(\frac{M_4(\lambda_C)}{M_4(\lambda)g(\lambda)}\right)^{1/2} \tag{6.21}$$

and $\bar{u}$, $\hat{h}$ remain unchanged.

We note that

$$\bar{K}(\lambda_C) = \frac{a(\lambda_C)t(\lambda_C)}{g(\lambda_C)^{1/2}}. \tag{6.22}$$

Assuming that $a(\lambda_C)/g(\lambda_C)^{1/2}$ is not far from unity, we can choose $t(\lambda_C)$ close to unity so that

$$\bar{K}(\lambda_C) = \frac{a(\lambda_C)t(\lambda_C)}{g(\lambda_C)^{1/2}} = 1. \tag{6.23}$$

This fixes $\bar{A}(\lambda_C)$ uniquely. By choosing $t(\lambda)$ judiciously it is possible to keep $\bar{A}(\lambda)$ fixed at $\bar{A}(\lambda_C)$. This defines $t(\lambda)$, and forces $\bar{K}(\lambda)$ to have a particular dependence in λ. None of the answers should depend on t, but the approximation is obviously better when $t \approx 1$. Should the calculation give t_C significantly different from unity, it is better to redefine $\bar{K}(\lambda_C)$ so that $t_C \approx 1$ again. We shall comment on this later.

The final step is to ignore the terms $O(J\chi^3)$ and $O(J^2)$ in the source term $[A(\chi+J) - A(\chi)]$ in (6.10) near the critical region and hence to keep just the linear term $\Sigma_k \hat{h}_k S_k$ in h after the change of variables has been made. Although difficult to justify fully, this assumption is generally adopted in a field theoretical approach to critical phenomena (Brézin *et al* 1976, Wegner 1976 and Brout 1974). We shall see later that it is justified empirically.

Given this assumption, the generating functional in $\bar{\chi}$ field in (6.10) takes the form

$$Z[h] = N\int \prod_i \mathrm{d}\bar{\chi}_i \exp\left(-\mathscr{L}[\bar{\chi}] + \sum_i \bar{\chi}_i h_i \mu(\lambda)\right) \tag{6.24}$$

where $\mathscr{L}[\bar{\chi}]$ is of the form (6.13) and $\mu(\lambda)$ is an appropriate renormalization factor.

Rescale $\bar{\chi}$ by $\hat{\chi}_i = \bar{\chi}_i \mu(\lambda)$ and (6.13) becomes

$$\mathscr{L}[\hat{\chi}] = -\tfrac{1}{2}\sum \frac{a(\lambda)}{\mu^2(\lambda)} \hat{\chi}_i \hat{\chi}_j I_{ij} + \sum\left(\frac{a(\lambda)}{\mu^2(\lambda)} d + \frac{b(\lambda)}{\mu^2(\lambda)}\right)\hat{\chi}_j^2 + \sum_j \frac{g(\lambda)}{\mu^4(\lambda)}\frac{M_4(\lambda)}{4!}(2d)^4 \hat{\chi}_j^4 \tag{6.25}$$

where the term involving the source h_i remains $\Sigma_i h_i \hat{\chi}_i$.

We see that arguments from (6.13) onward remain equally valid for $a(\lambda)/\mu^2(\lambda)$, $b(\lambda)/\mu^2(\lambda)$ and $g(\lambda)/\mu^4(\lambda)$ as for $a(\lambda)$, $b(\lambda)$ and $g(\lambda)$, and we can choose $\mu(\lambda) = 1$, without any loss of generality.

To summarize, in the critical region we have argued that the generating functionals

$$Z[J]=\int \prod_k \mathrm{d}\sigma_k|\sigma_k| \exp\left(-A\sum_k \sigma_k^2 - u\sum_k \sigma_k^4 + \tfrac{1}{2}\sum_{l,m} \sigma_l I_{lm}\sigma_m + \sum_k J_k\sigma_k\right) \tag{6.26}$$

and

$$Z[\hat{h}]=\int \prod_k \mathrm{d}S_k \exp\left(-\bar{A}\sum_k S_k^2 - \bar{u}\sum_k S_k^4 + \tfrac{1}{2}\bar{K}\sum_{l,m} S_k I_l S_m + \sum_k \hat{h}_k S_k\right) \tag{6.27}$$

describe the same critical behaviour, subject to $\bar{K}(\lambda_C)=1$, etc, provided we relate $\hat{h}_k$ to J_k through (6.18).

We can therefore assume the known results for the canonical theory (6.27) to demonstrate that, in the vicinity of the critical region appropriate to a theory with cut-off Λ, the non-canonical system (6.26) also behaves like a cut-off canonical theory. In particular, it permits a phase with broken symmetry entirely induced by the change in measure, in which Λ, the particle mass m_R and the field expectation value $v=\langle\phi\rangle$ are related. Low-energy results are independent of Λ.

Before doing so, we observe that the model of (6.26) only depends on the single-site measure $\mu(\sigma)$ via its moments. Thus, had we begun with a canonical theory

$$Z_{\mathrm{can}}[J]=\int \prod_k \mathrm{d}\sigma_k \exp\left(-A\sum_k \sigma_k^2 - u\sum_k \sigma_k^4 + \tfrac{1}{2}\sum_{l,m} \sigma_l I_{lm}\sigma_m + \sum_k J_k\sigma_k\right) \tag{6.28}$$

instead of (6.26) we could have repeated the chain of arguments to create a new canonical theory $Z[\hat{h}]$ of the same form as (6.27) in which the moments I_{2n} are changed accordingly. In this case, reapplying the tactics of Lüscher and Weisz (1987, 1988a) to $Z[\hat{h}]$ should give the same results as applying them to $Z_{\mathrm{can}}[J]$. This provides a stringent check on the validity of our approximations in the vicinity of the critical region. That the results indeed match very closely is shown in figure 5, using the methods of the remainder of this paper. With this behind us we feel confident to tackle the non-canonical theory.

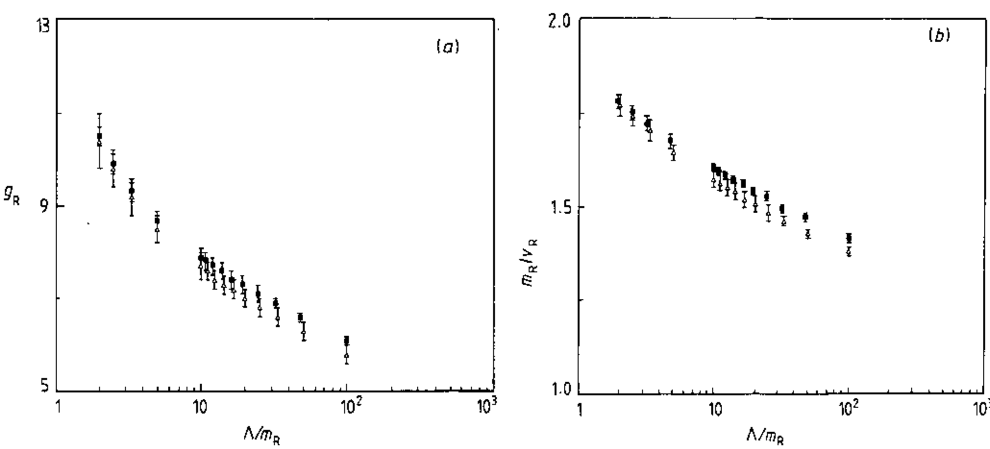

Figure 5. (*a*) Plot of g_R as a function of Λ/m_R in the symmetric phase of the canonical $\lambda\phi_4^4$ theory at $u=4.7699\times10^{-2}$, $A=1-2u$, chosen from [7]. (*b*) Plot of m_R/v_R as a function of Λ/m_R in the broken symmetric phase of the canonical $\lambda\phi_4^4$ theory at $u=4.7699\times10^{-2}$, $A=1-2u$, chosen from [7]. The squares denote results from our method and the triangles denote results of [7].

7. Comparing the σ- and S-systems

The spin correlations in the different systems are obtained by differentiating ln $Z[J]$ and ln $Z[\hat{h}]$ with respect to J_i and $\hat{h}_i$ respectively. Given the relationship (6.18) between $\hat{h}_i$ and J_i the correlation functions χ, χ_2, μ_2 are related as

$$\chi|_\sigma = I_2^2(\lambda)(2d)^2\left(\frac{M_4(\lambda_C)}{M_4(\lambda)g(\lambda)}\right)^{1/2}\chi\Big|_S \tag{7.1}$$

$$\chi_2|_\sigma = I_2^4(\lambda)(2d)^4\left(\frac{M_4(\lambda_C)}{M_4(\lambda)g(\lambda)}\right)\chi_2\Big|_S \tag{7.2}$$

$$\mu_2|_\sigma = \frac{I_2^2(\lambda)(2d)^2}{t(\lambda)}\left(\frac{M_4(\lambda_C)}{M_4(\lambda)g(\lambda)}\right)^{1/2}\mu_2\Big|_S. \tag{7.3}$$

The $\mu_2|_S$ contains the term $t(\lambda)$ since μ_2 contains the distance l between sites in its definition

$$\mu_2 = \sum_k \frac{l^2}{a^2}\langle\sigma_0\sigma_l\rangle_C \tag{7.4}$$

and the momentum scaling changes l without affecting a.

The renormalized mass m_R, coupling g_R and Z_R are defined in the usual way;

$$m_R = \left(\frac{8\chi}{\mu_2}\right)^{1/2} \tag{7.5}$$

$$g_R = -64\left(\frac{\chi_2}{\mu_2^2}\right) \tag{7.6}$$

$$Z_R = \frac{8\chi^2}{\mu_2} \tag{7.7}$$

whence

$$m_R|_\sigma = \sqrt{t(\lambda)}\, m_R|_S \tag{7.8}$$

$$g_R|_\sigma = t(\lambda)^2 g_R|_S \tag{7.9}$$

$$Z_R|_\sigma = t(\lambda)I_2^2(\lambda)\frac{(2d)^2}{g(\lambda)^{1/2}}\left(\frac{M_4(\lambda_C)}{M_4(\lambda)}\right)^{1/2} Z_R\Big|_S. \tag{7.10}$$

Near the critical point, we have the following correspondence between the parameter space of both systems:

$$(K = 1, A = d, \lambda = ua^{d-4}) \leftrightarrow (\bar{K}(\lambda), \bar{A}, \bar{u}). \tag{7.11}$$

In order to pin down the image of $(1, d, \lambda)$ in $(\bar{K}, \bar{A}, \bar{u})$ space we really need to consider

$$\frac{\chi_{(2)}|_{\sigma,\lambda_i}}{\chi^2|_{\sigma,\lambda_i}} = \frac{\chi_{(2)}|_{S,K_i}}{\chi^2|_{S,K_i}} \tag{7.12}$$

for a set of $\lambda_i \geqslant \lambda_C$, $\bar{K}_i < \bar{K}_C$ in the symmetric phase.

Remember that, for the σ system, the temperature K was fixed at unity. Thus, for $\lambda > \lambda_C$ we need to calculate χ, χ_2, etc slightly away from the critical region. To do this we evaluate the high-temperature series for the modified σ system

$$Z'[J] = \int \prod_k (\mathrm{d}\sigma_k|\sigma_k|)\exp(-A'\sum\sigma_k^2 - u'\sum\sigma_k^4 + \tfrac{1}{2}\sum\sigma_l I_{lm}\sigma_m + J_k\sigma_k) \tag{7.13}$$

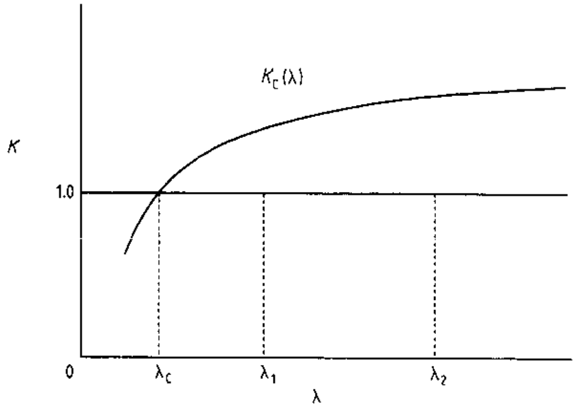

Figure 6. Plot of the critical temperature $K_C(\lambda)$ for the system (3.13). $K_C(\lambda_C)=1$ by definition.

where A', u', J are evaluated at $K=1$ (e.g. $A'=d$) but I_{lm} remains given in terms of an inverse temperature K.

The system (7.13) shows critical behaviour at an inverse temperature $K_C(\lambda)$ for which $K_C(\lambda_C)=1$ (figure 6). Ideally we should choose several points λ_i. In practice we choose a couple, $\lambda_1=7.5$ and $\lambda_2=8.0$, and assume linear interpolation in the vicinity of λ_C for the parameters $a(\lambda)$, $b(\lambda)$ and $g(\lambda)$.

We are now able to evaluate $\bar{K}$ as a function of λ in the vicinity of λ_C using Padé methods for the 11-term series of Baker and Kincaid (1981). The details of the calculation are given elsewhere (Wong 1990). The results are given in table 1.

8. Solution to the renormalization group equations in the symmetric phase

Given the initial value of $m_R|_S$, etc, in table 1 we can proceed to get closer to the critical point $\bar{K}_C=1$ through the S system RG equations. These determine the evolution of the S-system parameters as the critical point is approached at fixed $\bar{A}$, $\bar{u}$. Since we are dealing with a conventional quartic interaction we continue with the methods of Lüscher and Weisz (1987, 1988a), to which we refer the reader for details. Once this has been done the work of the previous section can be used to calculate the corresponding σ-system correlation functions $m_R|_\sigma$, etc. In this way we determine the behaviour of the σ-system in the symmetric region $\lambda>\lambda_C$.

In perturbation theory the β-function (and γ, δ functions) for the S-system is known to three loops. The crucial observation is that, since β is positive, the RG equation drives $g_R|_S$ to zero as $m_R|_S$ decreases and perturbation theory becomes an ever better approximation as we get closer to the critical line. Specifically, in the limit that $m_R|_S$ ($m_R|_\sigma$) goes to zero the coupling $g_R|_S$ goes to zero (as does $g_R|_\sigma$) according to the implicit asymptotic formula

$$m_R|_S=C_1(\bar{A},\bar{u})(\beta_1 g_R|_S)^{\beta_2/\beta_1^2}\exp(-1/\beta_1 g_R|_S)[1+O(g_R|_S)] \tag{8.1}$$

where $C_1(\bar{A},\bar{u})$ is a constant.

We also have

$$Z_R|_S=C_2(\bar{A},\bar{u})(1+O(g_R|_S)) \tag{8.2}$$

Table 1. Values of λ and t against $\bar{K}$ of the non-canonical theory as obtained by integrating the renormalization group equations in the symmetric phase of the complementary canonical theory.

$\bar{K}$	λ	$t(\lambda)$
0.9645 (1)	7.497 (1)	1.220 (16)
0.9763 (1)	7.128 (1)	1.205 (16)
0.9860 (1)	6.811 (1)	1.192 (16)
0.9933 (1)	6.561 (2)	1.181 (16)
0.9981 (1)	6.393 (2)	1.174 (16)
0.9984 (1)	6.381 (3)	1.174 (16)
0.9988 (1)	6.370 (3)	1.174 (16)
0.9990 (1)	6.360 (3)	1.173 (16)
0.9993 (1)	6.352 (3)	1.173 (16)
0.9994 (1)	6.344 (3)	1.172 (16)
0.9996 (1)	6.338 (3)	1.172 (16)
0.9998 (1)	6.333 (3)	1.172 (16)
0.9998 (1)	6.330 (3)	1.172 (16)
0.9998 (1)	6.327 (3)	1.172 (16)

Table 2. Values of $g_R|_\sigma$ and $Z_R|_\sigma$ against $m_R|_\sigma$ of the non-canonical theory as obtained by integrating the renormalization group equations in the symmetric phase of the complementary canonical theory. The corresponding values of $g_R|_S$, $m_R|_S$, $Z_R|_S$ and $Z_R^\Theta|_S$ are also given.

$m_R\|_S$	$m_R\|_\sigma$	$g_R\|_S$	$g_R\|_\sigma$	$Z_R\|_S$	$Z_R\|_\sigma$	$Z_R^\Theta\|_S$
0.48	0.53	26.7 (12)	39.8 (28)	1.0331 (1)	0.989 (39)	0.01726 (1)
0.39	0.42	23.9 (9)	34.6 (22)	1.0195 (1)	0.987 (39)	0.01796 (3)
0.29	0.32	21.1 (7)	29.9 (18)	1.0085 (3)	0.986 (39)	0.01876 (5)
0.19	0.21	18.3 (6)	25.5 (14)	1.0003 (4)	0.986 (39)	0.01973 (8)
0.10	0.11	15.0 (4)	20.7 (11)	0.9945 (5)	0.985 (39)	0.02114 (12)
0.09	0.098	14.6 (4)	20.1 (10)	0.9940 (5)	0.985 (38)	0.02134 (13)
0.08	0.087	14.2 (4)	19.5 (10)	0.9938 (5)	0.985 (38)	0.02157 (13)
0.07	0.076	13.7 (3)	18.9 (9)	0.9931 (5)	0.985 (38)	0.02181 (14)
0.06	0.065	13.2 (3)	18.2 (9)	0.9927 (5)	0.985 (38)	0.02209 (15)
0.05	0.054	12.7 (3)	17.5 (8)	0.9923 (5)	0.985 (38)	0.02241 (16)
0.04	0.043	12.1 (3)	16.7 (8)	0.9919 (5)	0.985 (38)	0.02279 (17)
0.03	0.033	11.4 (3)	15.7 (7)	0.9915 (5)	0.984 (38)	0.02327 (18)
0.02	0.022	10.6 (2)	14.5 (6)	0.9911 (6)	0.984 (38)	0.02391 (19)
0.01	0.011	9.4 (2)	12.9 (6)	0.9905 (6)	0.984 (38)	0.02494 (21)

and

$$Z_R^{(\Theta)}|_S = C_3(\bar{A}, \bar{u})(g_R|_S)^{-1/3}(1+O(g_R|_S)) \tag{8.3}$$

β_1 and β_2 are the one- and two-loop coefficients of the β-function.

By integrating the RG equations using the data of table 1 as initial values using numerical algorithms such as the Runge–Kutta method, we have evaluated $m_R|_S, g_R|_S, Z_R|_S$ and $Z_R^{(\Theta)}|_S$ for $\bar{K}$ up to $\bar{K}_C = 1$. The values are given in table 2. The corresponding results for the σ-system are also given there, on converting from $\bar{K}$ to λ. A plot of $g_R|_\sigma$ against $\Lambda/m_R|_\sigma$ is given in figure 7(a).

However, our main interest is in the broken symmetry phase, and it is to this that we now turn.

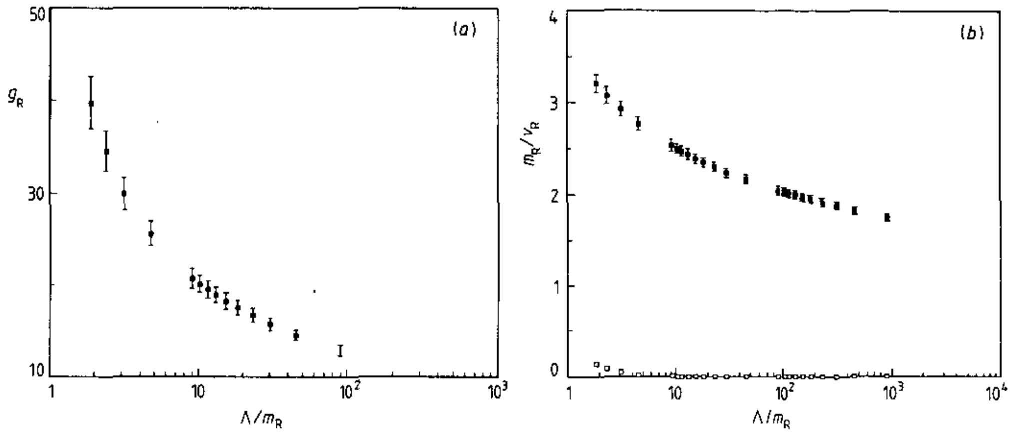

Figure 7. (a) Plot of g_R as a function of Λ/m_R in the symmetric phase of the non-canonical theory. (b) Plot of m_R/v_R as a function of Λ/m_R in the broken symmetric phase of the non-canonical theory. The full squares denote the renormalized results and the open squares denote the classical results (1.12).

9. Calculations in the broken symmetry phase

For $\lambda < \lambda_C$, $\bar{K} > \bar{K}_C$, the reflection symmetry $S \to -S$ of the action (6.26) is spontaneously broken and the field S acquires a non-zero expectation value

$$v|_S = \left.\frac{\partial \ln Z[\hat{h}]}{\partial \hat{h}}\right|_{\hat{h}=0} \tag{9.1}$$

whereas

$$v|_\sigma = \left.\frac{\partial \ln Z[J]}{\partial J}\right|_{J=0}. \tag{9.2}$$

They differ by a finite quantity.

Again we follow the definitions and renormalization scheme in Lüscher and Weisz (1987, 1988a). In particular the renormalized coupling is given by

$$g_R|_S = \frac{3m_R^2|_S}{v_R^2|_S} \tag{9.3}$$

where v_R is the renormalized vacuum expectation value of the scalar field. We can solve the Callan–Symanzik equation in the same way as in the symmetric phase, obtaining the asymptotic form of the scaling laws

$$m_R|_S = C_1'(\beta_1 g_R|_S)^{-\beta_2/\beta_1^2} \exp(-1/\beta_1 g_R|_S)[1+O(g_R|_S)] \tag{9.4}$$

where C_1' is an integration constant depending on $\bar{A}$, $\bar{u}$, and β_1, β_1 are the one- and two-loop coefficients of the β-function (identical to those in the symmetric phase). Similarly, we have the other scaling laws

$$Z_R|_S = C_2'[1+O(g_R|_S)] \tag{9.5}$$

$$Z_R^\Theta|_S = C_3' g_R|_S^{-1/3}[1+O(g_R|_S)] \tag{9.6}$$

$$(\bar{K}_C - \bar{K}) = C_3' g_R|_S^{-1/3} m_R^2[1+O(g_R|_S)]. \tag{9.7}$$

The C_i' are not quite the same as the C_i in (8.1)–(8.3), determined in the last section. However, relations between C_i and C_i' can be established since both the C_i' and C_i are defined at the critical line and thus they both can be given an interpretation in terms of the single critical massless theory.

Without repeating the arguments of Lüscher *et al* (1987, 1988a), it turns out that for our (and their) choice of renormalization conditions

$$C_1'(\bar{A}, \bar{u}) = e^{1/6} C_1(\bar{A}, \bar{u}) \tag{9.8}$$

$$C_i'(\bar{A}, \bar{u}) = C_i(\bar{A}, \bar{u}) \qquad i = 2, 3. \tag{9.9}$$

Now since each $C_i(\bar{A}, \bar{u})$ has been determined to a reasonable accuracy from the calculation in the symmetric phase, $C_i'(\bar{A}, \bar{u})$ is thus determined by equations (9.8), (9.9) and may be used as initial data for the integration of the renormalization group equations along fixed $(\bar{A}, \bar{u})$ in the broken symmetry phase starting at $\bar{K} = \bar{K}_C = 1$. Large values of $\bar{K}$ are reached by integrating the Callan–Symanzik equations numerically using the three-loop formulae for β, γ and δ in Lüscher *et al* (1987, 1988a) and we terminate the integration when $m_R|_S$ reaches 0.5 ($m_R|_{\sigma} = 0.533$).

We note that, although in principle the upper limit would be $m_R|_{\sigma} = 1$ (or written in explicit lattice units $m_R|_{\sigma} = \Lambda$), when $m_R|_S > 0.5$ we expect the $O(m_R^2 \ln m_R)$ contributions in the β (γ and δ) function to become significant. However, the $O(m_R^2 \ln m_R)$ contributions are non-universal (ambiguous) and dependent on the lattice chosen. In Lüscher *et al* (1987, 1988a), this ambiguity is demonstrated to be an intrinsic property of the renormalized vertex functions and thus unavoidable.

The $m_R|_{\sigma}$ etc are calculable from $m_R|_S$, etc, via the relations (7.1)–(7.3). The results of the integration are summarized in table 3. Here we see that the qualitative scaling behaviour in both the non-canonical and canonical systems is very similar.

More relevantly to our initial aims, in table 3 we note that for $m_R|_S \simeq 0.5$, $g_R|_S \simeq 23$ is less than a half of the tree unitary bound $g_R|_S \simeq 47$. Hence perturbation theory remains valid. As the scattering length is positive, the long-distance force is attractive and bound states may exist. A rough estimate of the threshold is given in Lüscher *et al* (1987, 1988a) to be $g_R|_S = 35$. Thus for a cut-off $\Lambda \geq 10 m_R$, bound states are unlikely to be present, while in the range $2m_R|_S < \Lambda < 10 m_R|_S$, bound states may exist but their binding energy should be rather small. Given the assumptions and approximations made in this work, the solutions obtained here should be checked by other more direct methods such as the Monte Carlo simulation (Kuti *et al* 1988). In particular, the value of $t_C \approx 1.17$, which makes the approximation of neglecting short wavelengths more suspect. In fact, the curve in figure 7(*b*) is expected to be higher than the true result in the same way as that for the canonical theory in figure 5(*b*). Redefining $\bar{K}(\lambda_C)$ so

Table 3. Values of $g_R|_\sigma$, $Z_R|_\sigma$, λ, t against $m_R|_\sigma$ of the non-canonical theory as obtained by integrating the renormalization group equations in the broken symmetric phase of the complementary canonical theory. The corresponding values of $g_R|_S$, $m_R|_S$, $Z_R|_S$, $Z^\ominus_R|_S$ and $\bar{K}$ are also given.

$m_R\|_S$	$m_R\|_\sigma$	$g_R\|_S$	$g_R\|_\sigma$	$Z_R\|_S$	$Z_R\|_\sigma$	$Z^\ominus_R\|_S$	$\bar{K}$	λ	$t(\lambda)$
0.01	0.011	9.1 (2)	12.5 (6)	0.9856 (7)	0.979 (38)	0.02523 (23)	1.0000 (1)	6.320 (1)	1.171 (16)
0.02	0.022	10.2 (2)	14.0 (6)	0.9837 (8)	0.977 (38)	0.02425 (20)	1.0003 (1)	6.315 (1)	1.171 (16)
0.03	0.033	11.0 (2)	15.0 (7)	0.9824 (8)	0.976 (38)	0.02365 (19)	1.0004 (1)	6.309 (1)	1.171 (16)
0.04	0.043	11.6 (2)	15.9 (7)	0.9813 (9)	0.975 (38)	0.02321 (18)	1.0006 (1)	6.302 (1)	1.170 (16)
0.05	0.054	12.1 (2)	16.6 (8)	0.9803 (9)	0.975 (38)	0.02286 (18)	1.0008 (1)	6.294 (1)	1.170 (16)
0.06	0.065	12.6 (3)	17.2 (8)	0.9794 (9)	0.974 (38)	0.02256 (16)	1.0010 (1)	6.286 (1)	1.170 (16)
0.07	0.076	13.0 (3)	17.8 (9)	0.9785 (10)	0.973 (38)	0.02231 (16)	1.0013 (1)	6.277 (1)	1.170 (16)
0.08	0.087	13.4 (3)	18.3 (9)	0.9778 (10)	0.973 (38)	0.02208 (15)	1.0015 (1)	6.268 (1)	1.169 (16)
0.09	0.097	13.8 (3)	18.8 (9)	0.9770 (10)	0.972 (38)	0.02188 (14)	1.0018 (1)	6.259 (1)	1.169 (16)
0.10	0.108	14.1 (3)	19.3 (10)	0.9763 (10)	0.972 (38)	0.02170 (14)	1.0021 (1)	6.248 (1)	1.168 (16)
0.20	0.22	16.9 (5)	22.9 (12)	0.9694 (13)	0.969 (37)	0.02045 (10)	1.0055 (1)	6.120 (1)	1.163 (16)
0.30	0.32	19.3 (6)	25.8 (15)	0.9625 (16)	0.968 (37)	0.02969 (9)	1.0100 (1)	5.949 (1)	1.156 (16)
0.40	0.43	21.5 (7)	28.4 (17)	0.9553 (18)	0.968 (37)	0.01913 (6)	1.0153 (1)	5.738 (1)	1.147 (16)
0.50	0.53	23.8 (9)	30.8 (20)	0.9476 (20)	0.968 (36)	0.01869 (4)	1.0214 (1)	5.483 (1)	1.137 (16)

that $t_C \approx 1$ lowers this curve, and gives a more reliable result. However, the qualitative conclusions are unchanged and, giving the messiness of the calculations, we do not repeat them here. However, the contrast in figure 7(*b*) between the renormalized and the 'classical result' (5.3) is striking. In figure 7(*b*) we have no difficulty in simultaneously enforcing $\Lambda/m_R|_\sigma \gg 1$, $\Lambda/v_R|_\sigma \gg 1$.

10. Conclusions

In the earlier sections of this paper we showed (numerically) that the continuum non-canonical theory of a single real field was in the same universality class as its canonical counterpart. In the latter part of the paper we argued for a much stronger link between the non-canonical and canonical cases, rewriting the non-canonical theory as a 'mean field' canonical theory.

Under fairly general assumptions we have established a correspondence between the two theories *near* the critical point. The scaling behaviour in both phases of the non-canonical system can then be determined through the scaling behaviour of the canonical system.

With the substantiation of the conjecture of de Alfaro *et al* in mind, our interest is in the broken symmetry phase of the cut-off theory (cut-off Λ). Just as in the canonical model examined by Lüscher and Weisz (1987, 1988a), from whom we have borrowed the tactics, we obtain an upper bound $m_R \leqslant 3.0 v_R$ on requiring $\Lambda \geqslant 2m_R$. (Here m_R is the renormalized mass and v_R the renormalized vacuum expectation value.) However, since the original conformally-invariant model has only one bare coupling, fixing any two parameters amongst m_R, g_R, Z_R, λ and Λ will determine values of all the others. For example $v_R = 250$ GeV, $m_R = 500$ GeV implies a cut-off Λ of 60 (10) Tev where classically $\Lambda = 250$ GeV. A plot of m_R/v_R against Λ/m_R is given in figure 7(*b*). Unlike the case in Lüscher and Weisz (1987, 1988a) the curve does not denote an upper bound, but a curve in which the theory lies. We see that the cut-off Λ is very much larger than that suggested from classical considerations (see, e.g. (5.3)). Had this not been so, the scheme would not have been sensible.

The scheme by Lüscher and Weisz that we have borrowed from so liberally was developed to calculate the Higgs mass bound. Both the canonical linear σ-model and the minimal Higgs mechanism suffer from an arbitrariness that the non-canonically quantized conformally invariant theory has managed to escape while still providing spontaneous symmetry breaking. (Although it could be argued that we only require a scale covariant measure, rather than a conformally invariant measure. This allows a further parameter B to be introduced that characterizes the measure.) It is not inconceivable that this idea may find real physical application in the above situations.

In conclusion, despite the unusual path integral representation, the scheme proposed by de Alfaro, Fubini and Furlan to quantize the conformally invariant scalar field theory does fulfil all its semiclassical promises after taking in the full quantum effects. The implications for a similar phenomenological quantum gravity theory are not unfavourable.

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